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SOME REMARKS ON THE RELATIVISTIC CONCEPTS OF VELOCITIES AND ACCELERATIONS*

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The current situation in the kinematics of four-dimensional Riemannian spaces has inspired the enunciation of the following comments and the generalization of the local Coriolis Theorem to the case of deformed media, in the framework of both the special and general theory of relativity.

1. Coordinates, frames of reference and world lines. The basic method in theoretical research is invariably associated with the use of coordinate systems: each point of a four-dimensional space is assigned four numbers (arithmetization), endowed with various postulated geometric properties, usually along with certain other formulated properties of a general nature. These tools make it possible to use the methods and operations developed in mathematical analysis.

Coordinate systems x^i or ξ^i $(i = 1, 2, 3, 4)$ in a given fixed space are generally associated with functions of the type

$$x^i = x^i(\xi^k) \quad \text{and} \quad \xi^k = \xi^k(x^i); k, i = 1, 2, 3, 4 \tag{1.1}$$

At a given set of points in the space one can consider the coordinate lines, declaring one of the coordinates, say, in systems ξ^i and x^k, ξ^4 and x^4, to be timelike and the coordinates ξ^α and x^α $(\alpha = 1, 2, 3)$ to be fully equivalent spacelike variables.

In applications, considering the same set of points, one can call one of the coordinate systems, say x^i, the observer's system; the other system ξ^k, with variables $\xi^\alpha = \text{const}$ and $\xi^4 = \tau$, is considered as a comoving Lagrangian frame in which the constant numerical values $\xi^\alpha = \text{const}$ name the points forming the world lines L, on which the successive positions of the individual points thus specified are determined by the timelike coordinate $\xi^4 = \tau$.

By the general formula (1.1), every point with fixed values of the three coordinates ξ^α determines laws of motion of the individual points, represented by a family of world lines L. Similarly, one can introduce families of world lines in Lagrangian variables x^4 and $x^\alpha = \text{const}$, for a law of motion - in the system of functions $\xi^i = \psi^i(x^\alpha, \tau)$. On a family of world lines for L, which are either given or to be determined for the particular continuum being studied, one considers vector elements $d\mathbf{r}$ of arbitrary families of world lines L; in Riemannian spaces one can put $|d\mathbf{r}| = ds$ and, knowing the functions (1.1), introduce an invariant metric axiomatically by the invariant formulae

$$ds^2 = \tilde{g}_{ij} dx^i dx^j = g_{ij} d\xi^i d\xi^j \tag{1.2}$$

Here $\tilde{g}_{ij}$ and g_{ij} are the components of the metric tensor, which depend only on the coordinates x^i or ξ^i and may be specified initially in either coordinate system; once that has been done they are uniquely defined in the other coordinate system on the basis of the functions (1.1), and define a corresponding Riemannian space.

2. Concepts of velocities. Model physical characteristics of moving points in four-dimensional spaces are generally introduced as individualized points, through the use of Lagrangian coordinates. In this way one obtains local and global concepts of comoving world lines L, proper time τ along L, arc length s along L and four-dimensional velocity $\mathbf{u} = d\mathbf{s}/(d\tau)$, which is directed along the tangent to L.

The fundamental geometric constant in the four-dimensional pseudo-Riemannian spaces of the special and general theory of relativity is a dimensional scalar constant, usually denoted by c (the three-dimensional velocity of light in a vacuum). In a special system of measurement units one may legitimately assume that $c = 1$. If $ds > 0$ on a world line L, one assumes furthermore that $d\tau > 0$, and therefore, if $c = 1$, one has $ds = d\tau$ and $|\mathbf{u}| = ds/d\tau = 1$ in the comoving coordinates. If the comoving world line is a null line, then $ds = d\tau = 0$ on L. Null comoving world lines correspond, in particular, to the motion of photons in a vacuum at a three-dimensional velocity $v = c$ in any local or global reference frame.

In the general case the corresponding three-dimensional velocity vector is defined by

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the formula $\mathbf{v} = d\mathbf{l}/d\tau$, where the infinitesimal vector $d\mathbf{l}$ is the "space" component of the vector ds appearing in the definition of the four-dimensional velocity vector $\mathbf{u} = ds/d\tau$ at the points of the world line L, directed along the tangent to L.

3. ***Universal canonical form of the metric in Lagrangian coordinates.*** In the general case of non-null world lines L in any Riemannian space, in both special and general relativity, one can find a global transformation of the type (1.1) from coordinates x^i to Lagrangian coordinates $\xi^1, \xi^2, \xi^3, \tau$, in which any metric takes the canonical form

$$ds^2 = c^2 d\tau^2 + 2c g_{\alpha 4} d\xi^\alpha d\tau + g_{\alpha\beta} d\xi^\alpha d\xi^\beta \tag{3.1}$$

In other words, one can make the component $g_{44}(\xi^\alpha, \tau)$ equal to c^2 or unity, thus introducing the variable coordinate τ in the entire space; at each point of the four-dimensional space one then specifies only nine components $g_{\alpha 4}(\xi^\alpha, \tau)$ and $g_{\alpha\beta}(\xi^\alpha, \tau)$, of the metric, and it is the choice of these components that determines the pseudo-Riemannian space. One example of such a transformation is

$$x^1 = \xi^1, \quad x^2 = \xi^2, \quad x^3 = \xi^3 \tag{3.2}$$

$$x^4 = \int_{\xi_0^4}^{\xi^4} [\tilde{g}_{44}(\xi^\alpha, \xi^4)]^{1/2} d\xi^4 = \varphi(\xi^\alpha, \xi^4, \xi_0^4)$$

After the transformation (3.1) we get

$$g_{\alpha 4} = \tilde{g}_{\alpha 4} - \frac{\partial\varphi}{\partial\xi^\alpha}, \quad g_{\alpha\beta} = \tilde{g}_{\alpha\beta} + \frac{\partial\varphi}{\partial\xi^\alpha}\frac{\partial\varphi}{\partial\xi^\beta} - $$

$$- \frac{\tilde{g}_{\alpha 4}}{\sqrt{\tilde{g}_{44}}}\frac{\partial\varphi}{\partial\xi^\beta} - \frac{\tilde{g}_{\beta 4}}{\sqrt{\tilde{g}_{44}}}\frac{\partial\varphi}{\partial\xi^\alpha}$$

The value of the variable coordinate $\tau = \xi^4$, which is defined at all points of the space, in the canonical form (3.1) is related to the choice of the family of transformations (1.1), which, as we have seen, represent the laws of motion of individual points $\xi^\alpha = \text{const}$ in Lagrangian form expressed as (1.1) with Lagrangian coordinates ξ^α, τ. In this way, for any family of world lines L, once the canonical form (3.1) of the metric has been chosen one obtains the concept of global time corresponding to the family L.

It is easy to see that the canonical form of the metric in the comoving Lagrangian coordinates ξ^α, τ is preserved under transformation (1.1) of the special type

$$\tau' = \tau + \psi(\xi^\alpha) \text{and} \xi'^\alpha = f^\alpha(\xi^\beta) \tag{3.3}$$

On such lines L and L' we have $d\tau' = d\tau$ for $\xi^\alpha = \text{const}$ and $\xi'^\alpha = \text{const}$, respectively; moreover, $g_{44}' = g_{44} = 1$ and the line L is transformed into L'. The components $g_{\alpha 4}$ and $g_{\alpha\beta}$ of the metric tensor and the tetrads of basis vectors $\mathbf{э}_i$ and $\mathbf{э}_i'$ remain invariant.

Thus, the metric of a pseudo-Riemannian space may always be reduced globally to the form (3.1), where ξ^α and τ denote the four coordinates which, since the metric is pseudo-Riemannian, are essentially non-equivalent, contrary to what is currently claimed in the literature.

For an arbitrary given family of world lines, τ is one of four global coordinates, the algebraic meaning of which is directly bound up with the form of the space metric.

Only when considering families L of coordinate world lines on which $\xi^\alpha = \text{const}$, so that also $ds^2 = c^2 d\tau^2$, can one assign the variable coordinate τ the sense of proper time on such generally arbitrary coordinate world lines with equations $\xi^\alpha = \text{const}$, $\alpha = 1, 2, 3$; in particular, these lines may be geodesics.

In the same fixed space one can consider metrics of type (3.1) for different families L and accordingly for different global times τ.

If L_1 and L_2 are two families which can be derived from one another by a coordinate transformation (3.3), then at corresponding points one has

$$d\tau_1 = d\tau_2$$

If the transformation is made from the observer's family L_2 with global time τ_2 to a family L_1 with proper global time τ_1, then at points M of intersection of L_1 and L_2 one obtains

$$d\tau_1 \neq d\tau_2$$

In the general case, considering the metric at such points M, and assuming that the tetrads are non-holonomically defined, one can write

$$\frac{ds_1^2}{c^2} = d\tau_1^2 - \left|\frac{dl_1}{d\tau_1}\right|^2 d\tau_1^2 \text{ and } \frac{ds_2^2}{c^2} = d\tau_2^2 - \left|\frac{dl_2}{d\tau_2}\right|^2 d\tau_2^2$$

where ds_1 and ds_2 correspond to a fixed space but to different families L and corresponding $d\tau$.

If $d\tau_1$ is the increment of proper time on L_1, then $\mathbf{v}_1 = d\mathbf{l}_1/d\tau_1 = 0$, but on L_2 we have $c^{-1}d\mathbf{l}_2/d\tau_2 = \mathbf{v}_2/c \neq 0$. In dimensional form we obtain

$$d\tau_1 = d\tau_2 \sqrt{1 - |\mathbf{v}_2|^2/c^2}$$

Here $d\tau_1$ is the proper time element on the world line L_1 and $d\tau_2$ is the corresponding time element on the observer's line L_2. If the subscripts 1 and 2 are interchanged, the observer's time τ_2 may be reckoned to be proper, while the time τ_1' on L_1 becomes the time of an observer on L_1.

For a fixed space and a corresponding fixed general situation, depending on different equally legitimate points of view, certain invariant inequalities hold both theoretically and experimentally: either $d\tau_1 < d\tau_2$ or $d\tau_2 < d\tau_1'$ on the fixed world lines L_1 and L_2, and these inequalities remain valid irrespective of the direction of the three-dimensional velocities $\mathbf{v}_1$ or $\mathbf{v}_2$.

Not infrequently, certain timelike coordinates in the general formula (1.2) for the metric are denoted by the letter t and treated as time; this interpretation is possible if the metric (1.2) can be reduced, by a suitable coordinate transformation, to the form (3.1) and one can identify τ and t. It is well-known, for example, that in the same space but different coordinates, the metric of the Schwarzschild field or that of the Lemaître field of the same Riemannian space can be written as

$$ds^2 = (1 - r_g/r)\, c^2dt^2 - (1 - r_g/r)^{-1}dr^2 - r^2d\Omega = c^2d\tau^2 - (r_g/r)\, dR^2 - r^2d\Omega$$

where

$$r = [{}^3/_2\,(R - c\tau)]^{2/3} r_g^{1/3},\ d\Omega = d\theta^2 + \sin^2\theta d\varphi^2$$

Clearly, both metrics determine the same system of geodesics. In that case, for certain geodesics - planetary orbits - one can fix $r = \text{const}$; then, in particular, we obtain

$$\tau = kt,\ k = (1 - r_g/r)^{1/2} = \text{const}$$

On the other hand, apart from the planetary orbits there are geodesics that pass by the horizon of a black hole or intersect a black hole. Any such geodesic may be viewed as the world line of a small particle in the Schwarzschild or Lemaître metric, with the corresponding coordinate τ. The variable t cannot be interpreted as time along a geodesic which intersects the horizon of a black hole. A simple analysis reveals that in the Lemaître metric the proper time τ on a geodesic intersecting the horizon of a black hole is such that $\tau_1 - \tau_0$ is finite, where τ_0 is the recorded time at some point on the geodesic and τ_1 is the time at the point of intersection of the geodesic with the horizon of the black hole.

The argument proposed here deepens our understanding of the time concept, and at the same time brings out the error in some monograph and textbook authors' assertions about the time necessary for a particle to fall into a black hole*. (*It may be useful to observe that the envelope systems of geodesic world lines cannot be viewed as geodesic world lines of individually defined points.)

4. Tetrads of basis vectors. Once coordinates and basis vectors have been introduced at each point of the space, one obtains invariant equalities for any infinitesimal vector $d\mathbf{r}$ with contravariant components dx^i, dy^i or $d\xi^i$, in any coordinate system:

$$d\mathbf{r} = dx^i\mathbf{э}_i' = dy^i\mathbf{э}_i'' = d\xi^i\mathbf{э}_i \tag{4.1}$$

with summation over the index $i = 1, 2, 3, 4$. In this situation, in the comoving metric (3.1) along L, when $d\xi^\alpha = 0$, we have

$$d\mathbf{r} = d\mathbf{s} = cd\tau\mathbf{э}_4 \tag{4.2}$$

and consequently, along L, it follows from (3.1) that $d\mathbf{r} = d\mathbf{s}$, and if $c = 1$ we have $|d\mathbf{r}| = |d\mathbf{s}| = d\tau$.

It follows from formulae (4.1), (1.1) and (3.1) that $g_{ij}(x^k) = \mathbf{э}_i'\mathbf{э}_j'$, $g_{ij}''(y^k) = \mathbf{э}_i''\mathbf{э}_j''$ and on L by (3.1) $g_{44} = 1$ when $c = 1$. In addition, the formulae for the tensor transformation of the basis vectors yield transformation formulae for the components g_{ij} of the metric tensor in a fixed Riemannian space. The specification of basis tetrads at each point of a Riemmanian space, in different global coordinate systems, determines a local metric. In a fixed Riemannian space the field of basis vectors is not generally arbitrary, since, for example, the components of the metric tensor of a Riemannian space form differential invariants, which are independent of the choice of coordinate system. Nevertheless, in general Riemannian spaces one can introduce any characteristics along world lines L, by using special systems of non-holonomic local basis tetrads at the points of the space.

Below we shall show that different observers, introducing different non-holonomic tetrads along a world line L, may accordingly define different acceleration vectors along L.

5. Possible specifications of arbitrary systems of non-holonomic tetrads. Let us assume that in any space, in an arbitrary basis $\mathbf{ э}_i$ a vector **A** is defined at every point relative to a holonomic or non-holonomic system of bases $\mathbf{э}_i$. Consider linear coordinate transformations relative to that system at the points of the space: the transformation from x^i to y^k, with coefficients $\mathscr{L} = \| a_{\cdot i}{}^{k\cdot} \|$, and the inverse transformation $\mathscr{L}^{-1} = \| b_{\cdot j}{}^{i\cdot} \|$, from y^k to x_i, so that $a^{k\cdot}{}_{\cdot i} b^{i\cdot}{}_j = \delta_j{}^k$; the coefficients $a_{\cdot i}{}^{k\cdot}$ may be arbitrary functions of ξ^α, τ and possibly other variables. Since **A** and $\mathbf{э}_i$ are all vectors, we can write

$$\mathbf{A} = A^i \mathbf{э}_i = A^i a_l{}^k b_i{}^l \mathbf{э}_k = A'^k \mathbf{э}_k'$$

In this connection, it is obvious that, if the transformations are non-degenerate, one can always assume that at each point of the space suitable linear algebraic transformations will replace the non-holonomic bases by orthonormal bases $\mathbf{e}_i$ and $\mathbf{e}^i$ with $\mathbf{e}_i \mathbf{e}^i = 1$ and $\mathbf{e}^i \mathbf{e}_j = 0$ for $i \neq j$. This is certainly the case if the original components of the basis vectors are contravariant and the components of the vector **A** are covariant.

At the same time, it is also clear that if the vector **A** and bases $\mathbf{э}_i$ depend not only on the coordinates ξ^α but also on scalar variables μ, then the derivatives $\partial \mathbf{A}/\partial \mu$ are vectors and the derivatives $\partial \mathbf{э}_i/\partial \mu$ are also vectors.

The proper global time τ in the metric (3.1), for a family of world lines L in the framework of the metric (3.1) and transformations (1.1), forms a scalar field for τ, while the three components $g_{\alpha 4}(\xi^\alpha, \tau)$ form a vector field; together these fields characterize the Riemannian space and the family of lines L. These facts together with the theory of accelerations proposed below, represent an important mechanical interpretation of the properties of Riemannian spaces as represented by the canonical form of the metric (3.1) in Lagrangian variables.

Orthonormal bases, as they vary in space and along the world lines L, are convenient representations of invariant properties and characteristics of observers and enable one to gain a clearer idea of the meaning of inertial observers; the latter may be defined at the points of L axiomatically, by stipulating local conditions $\mathbf{э}_i = \text{const}$ or $\mathbf{e}_i = \text{const}$, or, respectively, $\mathbf{э}^i = \text{const}$, assumed to hold simultaneously.

It follows from these locally possible definitions that the corresponding local inertial tetrads in curved Riemannian spaces cannot be holonomic, since the Riemann curvature tensor need not vanish. In the latter case it is impossible to introduce global inertial coordinates similar to Cartesian coordinates. Global inertial coordinates can be introduced only in Euclidean and Minkowski spaces.

6. Definition of the corresponding four-dimensional velocity and acceleration vectors. In the metric (3.1) the components of the vectors u satisfy the following formulae along comoving lines L:

$$u^4 = 1,\ u^\alpha = 0 \text{ and } u_4 = 1,\ u_\alpha = g_{\alpha 4}\ (\xi^\alpha, \tau) \tag{6.1}$$

and in bases $\mathbf{э}^k$ on the same lines L:

$$\mathbf{u} = \mathbf{э}_4 = \mathbf{э}^4 + g_{\alpha 4}(\xi^\alpha, \tau)\, \mathbf{э}^\alpha \tag{6.2}$$

Formula (6.2) holds in different four-dimensional Riemannian spaces, in which the field of four-dimensional velocities u defined for $\mathbf{u} = \mathbf{э}_4$ and the corresponding world lines L may form different families with a superimposed metric of type (3.1) in Lagrangian coordinates ξ^α, τ.

In this formulation of the problem, the partial derivative with respect to the proper global time

$$\partial \mathbf{u}/\partial \tau = \mathbf{a}_{\text{rel}} \tag{6.3}$$

yields the relative acceleration vector along the world lines L.

Since $\mathbf{u} = \mathbf{э}_4$ is a unit vector directed along the tangent to L, it follows that the acceleration vector $\mathbf{a}_{\text{rel}}$ is always perpendicular to the unit basis $\mathbf{э}_4 = \mathbf{u}$.

By formula (6.2), the relative acceleration vectors can be expressed in terms of the generally variable components of the metric $g_{k4}(\xi^\alpha, \tau)$ and the contravariant basis vectors $\mathbf{э}^k(\xi^\alpha, \tau)$, considered at points of world lines L, whether the latter are given or subject to determination.

The functions and world lines in (6.2) are related to one another, but they may be differently defined in any fixed Riemannian space. Accordingly, depending on $\mathbf{э}_k$ and $\mathbf{э}^k$ one may obtain different proper global timelike variables τ and accelerations $\mathbf{a}_{\text{rel}}$ for different families of world lines L.

The reduction of the metric to the form (3.1) is obviously aimed precisely at ensuring the existence of the proper global time τ occurring in the definition of velocities and accelerations in the special and general theories of relativity.

Moreover, the value of the vector component $g_{\alpha 4}$ in (3.1) is extremely important: it clarifies the physical meaning of suspected components of the four-dimensional vector $\mathbf{u}(\tau)$, which has covariant components $g_{\alpha 4}(\xi^\alpha, \tau) = u_\alpha$ relative to the contravariant bases in the tetrads $\mathfrak{s}^i$ in fixed pseudo-Riemannian spaces - these bases are generally variable along the world lines L.

By virtue of (6.1) and (6.2), in the comoving coordinates it may be necessary, at each point of the world line L, to use the selected bases $\mathfrak{s}_4 = \mathbf{u}$ and variables $g_{k4}(\xi^\alpha, \tau) = u_k(\xi^\alpha, \tau)$ locally as envelope, using different tetrad bases $\mathfrak{s}^k$, provided that Eq.(6.2) is satisfied.

Differentiating (6.2) with respect to τ along L in accordance with the account in the previous section, we obtain the following formulae in the holonomic basis $\hat{\mathfrak{s}}^i$ and the non-holonomic orthonormal basis $\mathbf{e}^i$

$$\frac{\partial \mathfrak{s}_4}{\partial \tau} = \hat{g}_{k4}\frac{\partial \hat{\mathfrak{s}}^k}{\partial \tau} + \frac{\partial \hat{g}_{k4}}{\partial \tau}\hat{\mathfrak{s}}^k = g_{k4}\frac{\partial \mathbf{e}^k}{\partial \tau} + \frac{\partial g_{k4}}{\partial \tau}\mathbf{e}^k \tag{6.4}$$

It is easy to see that in a metric of type (3.1) the following invariant relations hold at each point of a given world line L:

1) $\hat{g}_{44} = g_{44} = 1$, where by construction $\mathfrak{s}_4 = \mathbf{e}^4 = \mathfrak{s}^4 + g_{\alpha 4}\mathfrak{s}^\alpha$, provided that $\partial \mathfrak{s}_4/\partial\tau \neq \partial \mathbf{e}^4/\partial\tau$;

2) in veiw of the fixed nature of the Reimannian space,

$$\partial\hat{\mathfrak{s}}^k/\partial\tau = -\hat{\Gamma}^k_{4l}\hat{\mathfrak{s}}^l \text{ and } \partial\mathbf{e}^k/\partial\tau = -\Gamma^k_{4l}\mathbf{e}^l$$

and we therefore obtain

3) $\hat{g}_{k4}\partial\hat{\mathfrak{s}}^k/\partial\tau = g_{k4}\partial\mathbf{e}^k/\partial\tau = 0$.

From (6.4) we obtain the following formula for the acceleration:

$$\partial\hat{g}_{\alpha 4}/\partial\tau\hat{\mathfrak{s}}^\alpha = \partial g_{\alpha 4}/\partial\tau\mathbf{e}^\alpha$$

The acceleration is perpendicular to the vector $\mathfrak{s}_4$ as required, since $\mathfrak{s}_4 = \mathbf{e}^4$ and the vectors $\mathbf{e}^\alpha$ are perpendicular to $\mathbf{e}^4$ by orthonormality.

Thus, in the comoving Lagrangian system, the definition of the proper coordinate system on any line L, corresponding to the equations $\xi^\alpha = \text{const}$, implies the following formula for the relative acceleration:

$$\mathbf{a}_{\text{rel}} = \partial\hat{g}_{\alpha 4}/\partial\tau\mathfrak{s}^\alpha \quad \text{for} \quad \mathfrak{s}^\alpha \text{ generally variable.} \tag{6.5}$$

We emphasize that in the general case the following relations hold along L in the co-moving system:

$$ds^2 = c^2 d\tau^2,\ \mathbf{u} = \mathfrak{s}_4 \text{ and } g_{k4} = \mathfrak{s}_k\mathfrak{s}_4 \tag{6.6}$$

which may generally be postulated in advance*. (*Obviously, any families L, with a suitably superimposed metric at $g_{\alpha 4}(\xi^\alpha)$, may be considered as a family of geodesics; to that end one need only take the bases $\mathfrak{s}^k$ and components g_{k4} constant along L.)

In connection with Eq.(6.5), it is clear that the acceleration $\mathbf{a}_{\text{rel}}$ on L is defined by specifying the values of the components $g_{\alpha 4}$, which in turn are determined by specifying the basis tetrads $\mathfrak{s}_k$ and $\mathfrak{s}^k$ at the points of the world lines L.

In follows from (6.5) that the accleration $\mathbf{a}_{\text{rel}}$ depends essentially on the basis vectors in the observer's tetrads $\mathfrak{s}^k$ or $\mathfrak{s}_k$, which may be variable along L and are expressed holonomically or non-holonomically in terms of the components g_{k4}.

If all the bases $\mathfrak{s}_k$ and $\mathfrak{s}^k$ are constant, and therefore independent of the time τ, then the components $g_{\alpha 4}$ are also constant and therefore the acceleration is zero. In that case the corresponding lines L in (6.5) and (6.8) are geodesics.

$$\mathbf{a}_{\text{rel}} = \mathbf{a}_{\text{abs}} = 0 \tag{6.7}$$

If $g_{44} = 1$ in the metric (3.1), then $g_{\alpha 4}$ are non-zero and the usually non-holonomically defined components $g_{\alpha 4}$ are generally variable only through $\mathfrak{s}_4(\tau)$ when the bases $\mathfrak{s}_\alpha$ are constant.

In these cases, the corresponding acceleration on the world line L is said to be absolute. For absolute accelerations on a world line one can write

$$\mathbf{a}_{\text{rel}} = \mathbf{a}_{\text{abs}} = \frac{\partial g_{\alpha 4}(\xi^\alpha, \tau)}{\partial\tau}\mathfrak{s}^\alpha \text{ if } \mathfrak{s}^\alpha = \text{const} \tag{6.8}$$

In the general case one can assume that the bases $\mathfrak{s}_\alpha, \mathfrak{s}^\alpha$ or $\mathbf{e}^\alpha$ and $\mathbf{e}_\alpha$ form a non-holonomic orthonormal system of tetrads on L, making it possible to interpret the orthonormal system of tetrad vectors $\mathbf{e}_\alpha$ and their derivatives with respect to τ as characteristics of the observer's tetrads, resulting from his acceleration and three-dimensional tensors of rate of strain and rotation.

If the bases $\mathfrak{s}_\alpha$ or, respectively, $\mathfrak{s}^\alpha$ in the local tetrads are stipulated by definition to be constant with respect to time τ, such model tetrads are often introduced, and the corresponding observers with constant tetrad bases are said to be inertial. For inertial tetrads in comoving coordinates as in (6.2), the acceleration vector of a point on a line L is defined by formula (6.8), as in the case of an inertial observer. Inertial tetrads may be introduced axiomatically at each point of space, but the definition is unique only up to a Lorentz transformation in special and general relativity; this has no effect on the magnitudes of the acceleration vector.

Accelerations in relative inertial frames of reference, which are defined up to a Lorentz transformation in relativity theory and up to a Galilean transformation in Newtonian mechanics, are of particularly great physical importance.

Formulas (6.5) and (6.8) correspond to a comoving reference frame with a metric of type (3.1) for local generally non-holonomically defined moving tetrads, considered in a Riemannian space.

In a given space, the conversion of characteristics of motions from the comoving system to any other given reference system, considered in a general setting, is the central problem of the theory of inertial navigation*. (*Sedov L.I. and Tsypkin A.G., Elements of the Macroscopic Theories of Gravitation and Electromagnetism, Fizmatgiz, Moscow, 1989.)

In this connection it should be emphasized that for a given individual point the tetrads $\mathfrak{s}^k$ and world lines L in the theory of continuous media, whether in the framework of Newtonian or relativistic mechanics, may be arbitrary. However, in view of the equation of continuity, one should note that in a fixed Riemannian space, allowance for possible transformations of the metric tensor components implies that the volume distribution of the tetrads $\mathfrak{s}^i$ and the family of world lines L cannot be arbitrary, because of the differential equations of continuity, similar to the St. Venant equations in Newtonian mechanics and the Bianchi identities in general relativity.

Previously we established a relationship between $\mathbf{a}_{\text{rel}}$ and the derivatives with respect to τ of the observer's $\partial g_{\alpha 4}(\xi, \tau)$. In applications $\mathbf{a}_{\text{rel}}$ may be specified directly or subject to determination, based on additional dynamic or kinematic arguments or in terms of the derivatives $\partial g_{\alpha 4}/\partial\tau$, which in turn may be specified on the basis of equivalent mechanical reasoning about the given basis tetrads of the observer.

The sequence of bases $\mathfrak{s}^k(\tau)$ and $\mathbf{e}^k(\tau)$ on L, which defines the observer's system, may be specified continuously and arbitrarily, it may refer to different Riemannian spaces.

Obviously, if the components $g_{\alpha 4}$ vanish on the world lines L or depend there only on ξ^α, i.e., $g_{\alpha 4}(\xi^\alpha)$, then it follows from (6.5) that $\mathbf{a}_{\text{abs}} = 0$, and consequently the family of corresponding world lines L consists of geodesics, while if $g_{\alpha 4} = 0$ the global metric (3.1) in finite volumes of space has a synchronous form:

$$ds^2 = c^2 d\tau^2 + g_{\alpha\beta} d\xi^\alpha d\xi^\beta \tag{6.9}$$

When the metric is synchronous, the family of comoving lines consists of geodesics, since as $g_{\alpha 4} = 0$ the absolute accelerations vanish on all the world lines. Nevertheless, even in this case we cannot generally call a global family of world lines L an inertial family, as we did for inertial families of global coordinate lines $\xi^\alpha = \text{const}$ in Cartesian systems in Euclidean or Minkowski space. The reason is that in the general case of Riemannian spaces the second term of (6.9) contains, in particular, a three-dimensional matrix with components $g_{\alpha\beta}(\xi^\alpha, \tau)$, which cannot be diagonalized by a global transformation of the coordinates simultaneously at all points of the family of geodesic world lines L.

Various aspects of the construction of global synchronous reference systems in Riemannian spaces will be discussed elsewhere.

*7. **Generalization of the Coriolis formula to relativistic spacetime models in local theories of motion with accelerations, deformations and rotations.*** In the local three-space Σ, at points on a world line L, take some point M and consider there three infinitesimal space vectors, related through the equality

$$d\mathbf{r} = d\mathbf{r}_1 + d\mathbf{r}_2 \tag{7.1}$$

and let $d\tau$ be the element of global proper time on L at M, where all the increments $d\mathbf{r}$, $d\mathbf{r}_1$, $d\mathbf{r}_2$ are three-vectors. In this situation the local value of $d\tau$ is entirely analogous to the element of absolute time in Newtonian mechanics, but it is proper time in the framework of the comoving metric (3.1). After dividing (7.1) by $d\tau$ and proceeding as in Newtonian mechanics, one obtains the appropriately named three-dimensional velocities:

$$\mathbf{v}_{\text{abs}} = \mathbf{v}_{\text{tr}} + \mathbf{v}_{\text{rel}} \tag{7.2}$$

This equality will not hold if, instead of a uniform time with element $d\tau$ one introduces three proper times $d\tau_{abs}$, $d\tau_{tr}$, $d\tau_{rel}$. In the sequel, referring to absolute, translational and relative velocities we shall have in mind the velocity vectors occurring in Eq.(7.2). Assuming (7.2) to be true, we can consider its corollary when the acclerations for $d\mathbf{r}_{abs}$ and $d\mathbf{r}_{tr}$ are defined in the same frame of reference, e.g., an inertial frame, whereas the acceleration for $d\mathbf{r}_{rel}$ is defined in the observer's frame, which is frozen into the translational frame (e.g., in a liquid medium).

Thus, the absolute motion and translational motion of points may be defined relative to local inertial tetrads, while relative accelerations are defined in relation to the frozen tetrads of an observer moving along with the frozen tetrads of the reference frame, with allowance for accelerations, deformations and rotation of tetrads in translational motion. In classical Newtonian mechanics rigid bodies are almost automatically provided with a frozen translational frame of reference, and in that case one has the Coriolis formula.

An analogous formula for deformed frames of reference in translational motion in the Newtonian and relativistic theories, which is rather more complicated in form, can be derived by analogous arguments; the derivation may be extended to both special and general relativity*. (*More detailed derivations in curvilinear coordinates have been published in Sedov L.I., On the addition of motions relative to deformed reference systems. Prikl. Mat. Mekh., 42, 1, 175-177, 1978.)

Below we reproduce some necessary conclusions following from formula (7.2):

$$\mathbf{a}_{abs} = \frac{\partial \mathbf{v}(\xi^\alpha, \tau)}{\partial \tau} = \frac{\partial v^\alpha_{abs}}{\partial \tau}\mathbf{э}_\alpha + v^\beta_{abs}\nabla_\beta v^\alpha_{abs}\mathbf{э}_\alpha$$

This formula also holds for accelerations relative to a non-inertial observer, provided that the bases $\mathbf{э}_\alpha$ are replaced by bases corresponding to the observer's tetrad.

We now consider the different accelerations for a moving point M - the derivatives with respect to time τ of the different velocity vectors introduced previously.

The following derivation of the generalized Coriolis formula makes allowance for the arbitrary nature of translational motion, as in both Newtonian mechanics and (special and general) relativity theory.

Based on the definitions of velocity introduced in Eq.(7.1), let us consider reference frames in a local three-dimensional volume Σ on the world line L, at neighbouring points M and M' on L, corresponding to times τ and $\tau + d\tau$ in the limit as $d\tau \to 0$.

The coordinate bases are defined as follows.

1°. The basic local system of the observer (inertial by its very nature) with coordinates x^α and bases $\mathbf{э}_\alpha = \text{const}$ and with velocity $\mathbf{v}_{abs} = \partial\mathbf{r}/\partial\tau$, where τ is proper time along L.

2°. A moving translational system, rotating and deforming, with coordinates y^α and bases $\hat{\mathbf{э}}_\alpha$ which are variable with respect to $\mathbf{э}_\alpha$, with velocity $\mathbf{v}_{tr} = \partial\mathbf{r}_1/\partial\tau$ in the coordinate system x^α; the velocity vector has components $v_{tr}{}^\alpha$ in the $\mathbf{э}_\alpha$ bases and $\hat{v}_{tr}{}^\alpha$ in the $\hat{\mathbf{э}}^\alpha$ bases.

3°. Relative motion at velocity $\mathbf{v}_{rel} = \partial\mathbf{r}_2/\partial\tau = \hat{v}^\alpha\hat{\mathbf{э}}_\alpha$, whose components may be considered both in the tetrad $\hat{\mathbf{э}}_\alpha$ with components $\hat{v}^\alpha$ in $\hat{\mathbf{э}}_\alpha$ and in the tetrad $\mathbf{э}_\alpha$ with components $\mathbf{v}^\alpha_{rel}$. When the accelerations are computed, the results may be expressed in terms of any bases and assumed to coincide at any specific instant of time. For each acceleration, however, one must take into consideration that the derivatives of the basis vectors with respect to τ are different. For absolute acceleration in the basis $\mathbf{э}_\alpha$, we write

$$\mathbf{a}_{abs} = \frac{d^2\mathbf{r}(x^\alpha, \tau)}{\partial\tau^2} = \frac{dv^\alpha_{abs}\mathbf{э}_\alpha}{d\tau} = \left(\frac{\partial v^\alpha_{abs}}{\partial\tau} + v^\beta_{abs}\nabla_\beta v^\alpha_{abs}\right)\mathbf{э}_\alpha \tag{7.3}$$

The relations

$$v^\alpha_{rel} = \frac{\partial y^\alpha}{\partial\tau}, \quad \mathbf{a}_{rel} = \frac{d\hat{v}^\alpha_{rel}}{d\tau}\hat{\mathbf{э}}_\alpha + \hat{v}^\alpha_{rel}\frac{d\hat{\mathbf{э}}_\alpha}{d\tau} = \left(\frac{\partial\hat{\mathbf{v}}_{rel}}{\partial\tau} + \hat{v}^\beta_{rel}\nabla_\beta\hat{v}^\alpha_{rel}\right)\hat{\mathbf{э}}_\alpha + \hat{v}^\alpha_{rel}\frac{\partial\mathbf{v}_{tr}}{\partial y^\alpha} \tag{7.4}$$

yield the equality

$$\mathbf{a}_{rel} = a^\alpha_{rel}\hat{\mathbf{э}}_\alpha + \hat{v}^\alpha_{rel}\hat{\nabla}_\alpha\hat{v}^\beta_{tr}\hat{\mathbf{э}}_\beta$$

which may be rewritten, after transforming the second term, in the bases $\mathbf{э}_\beta$ as follows:

$$\mathbf{a}_{rel}\big|_{\mathbf{э}_\alpha=\text{const}} = \mathbf{a}_{rel}\big|_{\hat{\mathbf{э}}_\alpha=\text{const}} + v^\alpha(\nabla_\alpha v_{tr\,\beta})\mathbf{э}^\beta \tag{7.5}$$

Now, using (7.4), we can write

$$\mathbf{a}_{tr}|_{\mathbf{э}_\alpha=\text{const}} = \mathbf{a}_{tr}|_{\hat{\mathbf{э}}_\alpha=\text{const}} + \frac{\partial \mathbf{v}_{tr}}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial \tau}\Big|_{\hat{\mathbf{э}}_\alpha=\text{const}} + v^\alpha_{\text{rel}}(\nabla_\alpha v_{\text{tr}\,\beta})\,\mathbf{э}^\beta. \tag{7.6}$$

Differentiating (7.2) with respect to τ from the viewpoint of an observer with basis $\mathbf{э}_\alpha$ (assumed to be inertial), we find

$$\mathbf{a}_{\text{abs}} = \mathbf{a}_{tr}|_{\hat{\mathbf{э}}_\alpha=\text{const}} + \mathbf{a}_{\text{rel}}|_{\hat{\mathbf{э}}_\alpha=\text{const}} + 2v^\alpha_{\text{rel}}\nabla_\alpha v_{\text{tr}\,\beta}\mathbf{э}^\beta \tag{7.7}$$

Formula (7.7) is a generalization of the classical Coriolis formula, which is established to Newtonian mechanics and derived in Cartesian coordinate systems with orthonormal bases $\mathbf{э}_\alpha$ and $\hat{\mathbf{э}}_\alpha$.

Formula (7.7) holds in both special and general relativity theory, in any curvilinear coordinates.

If one uses the equality

$$\nabla_\alpha v_{\text{tr}\,\beta} = {}^1/_2(\nabla_\alpha v_\beta + \nabla_\beta v_\alpha) + {}^1/_2(\nabla_\alpha v_\beta - \nabla_\beta v_\alpha) = e_{\alpha\beta} + \omega_{\alpha\beta}$$

where $e_{\alpha\beta}$ are the components of the rate of strain and $\omega_{\alpha\beta}$ those of the rotation tensor in translational motion in the volume Σ, the generalized Coriolis formula (7.7) may be written as

$$\mathbf{a}_{\text{abs}} = \mathbf{a}_{tr} + \mathbf{a}_{\text{rel}} + 2v^\alpha_{\text{rel}}(e_{\alpha\beta} + \omega_{\alpha\beta})\,\mathbf{э}^\beta \tag{7.8}$$

where each term is represented in a tangible form.

Formula (7.8) retains its form when the accelerations are considered relative to a non-inertial observer with basis $\mathbf{э}_\alpha$.

Though based on the most elementary concepts of tensor analysis, the foregoing arguments provide a more general result in a more general situation, with practically no computations, at the same time demonstrating the reason for the appearance and nature of the "added" acceleration in formula (7.8).

The motion of a moving point M is split up into absolute and relative motions owing to the introduction of reference frames of translational motion; such frames may be introduced holonomically, together with global time, or locally - and in general non-holonomically - for each position of the moving point M.

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INVERSION OF LAGRANGE'S THEOREM FOR A RIGID BODY WITH A CAVITY CONTAINING A VISCOUS LIQUID*

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The stability of the state of equilibrium of a rigid body with a cavity partly or completely filled with a viscous incompressible liquid possessing surface tension is cosidered in a linear form. Lyapunov's direct method is used to show that the system is unstable if the second variation of the potential energy can take negative values. A priori lower and upper bounds for the solutions, when the perturbations are increased, are obtained. The lower bound guarantees exponential growth of the deviations of the solid and liquid particles from the equilibrium state. The upper bound shows that the solutions cannot increase at more than an exponential rate. In both cases the exponents are calculated from the parameters of the equilibrium state and the initial data for the perturbation fields.

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